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Characterization of Double Representation of Quaternion Quasi-Normal Matrices

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ABSTRACT

In this paper, the properties of quaternion quasi-normal matrices in the form of double representation of complex matrices. The sum and basic product of the quaternion matrices. Also this is applied to the double representation of quaternion matrices.

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Key words: quaternion quasi-normal, normal, quasi-normal.

INTRODUCTION

A normal matrix $A = (a_{ij})$ with complex elements is a matrix such that $AA^{CT} = A^{CT}A$ where A^{CT} denotes the (complex) conjugate transpose of A. In an article by K. Morita[4] a quasi-normal matrix is defined to be a complex matrix A which is such that $AA^{CT} = A^{T}A^{C}$, where T denotes the transpose of A and A^{C} the matrix in which each element is replaced by its conjugate, and certain basic properties of such a matrix are developed there.

In this paper, some basic results about to the quaternion quasi-normal are developed, which will need as to study the further development and application of quaternion quasi-normal matrices.

Theorem: 1[3]

If a quaternion matrix $A \in H_{n\times n}$ is defined as double representation of the form $A = A_0 + A_1 j$ where A_0 and A_1 are quasi-normal then A^C is also a quaternion quasi-normal.

Proof:

Let $A \in H_{n\times n}$ then we defined as $A = A_0 + A_1 j$ where A_0 and A_1 are quasi-normal. We have to prove that A^C is also a quaternion quasi-normal. $A = A_0 + A_1 j$ implies that $A^C = A_0^C - A_1^C j$

We have $A^C(A^C)^{CT} = (A_0^C - A_1^C j)(A_0^C - A_1^C j)^{CT} = A_0^C A_0^T - A_1^C A_1^T j$ $= (A_0^T A_0^C)^T - (A_1^T A_1^C)^T j$ [Since $A_0 A_0^{CT} = A_0^T A_0^C$ and $A_1 A_1^{CT} = A_1^T A_1^C$] $= (A_0^{CT} - A_1^{CT} j)(A_0 + A_1 j)$ $= (A^C)^T (A^C)^C$.

Theorem: 2

If a quaternion matrix $A \in H_{n\times n}$ is defined as double representation of the form $A = A_0 + A_1 j$ where A_0 and A_1 are quasi-normal then A^T and A^{CT} are also a quaternion quasi-normal.

Proof:

Let $A \in H_{n\times n}$ then we defined as $A = A_0 + A_1 j$ where A_0 and A_1 are quasi-normal. We have to prove that A^T is also a quaternion quasi-normal. That is $A^T (A^T)^{CT}$ = $(A^T)^T (A^T)^C$. $A = A_0 + A_1 j$ implies that $A^T = A_0^T + A_1^T j$. Now $A^T (A^T)^{CT}$ = $(A_0^T + A_1^T j)(A_0^C - A_1^C j)$ = $A_0 A_0^{CT} - A_1 A_1^{CT} j$ [Since $A_0 A_0^{CT} = A_0^T A_0^C$ and $A_1 A_1^{CT} = A_1^T A_1^C$] = $(A_0^T + A_1^T j)^T (A_0^T + A_1^T j)^C$ = $(A^T)^T (A^T)^C$.

Next we have to prove that A^{CT} is also a quaternion quasi-normal. That is $A^{CT} (A^{CT})^{CT}$ = $(A^{CT})^T (A^{CT})^C$. Let us consider $A = A_0 + A_1 j$ implies that $A^{CT} = A_0^{CT} - A_1^{CT} j$. Now, $A^{CT} (A^{CT})^{CT}$ = $(A_0^{CT} - A_1^{CT} j)(A_0 + A_1 j)$ = $A_0^T A_0^C - A_1^T A_1^C j$ [By the definition $AA^{CT} = A^{CT} A$ = $A^T A^C$] = $(A_0^C - A_1^C j)(A_0^T + A_1^T j)$ = $(A^{CT})^T (A^{CT})^C$.

Theorem: 3

If a quaternion matrix $A \in H_{n\times n}$ is defined as double representation of the form $A = A_0 + A_1 j$ where A_0 and A_1 are quasi-normal then A^{-1} and λA are also a quaternion quasi-normal.

Proof:

Let $A \in H_{n\times n}$ then we defined as $A = A_0 + A_1 j$ where A_0 and A_1 are quasi-normal. We have to prove that A^{-1} is also a quaternion quasi-normal. That is $A^{-1}(A^{-1})^{CT}$ = $(A^{-1})^T (A^{-1})^C$.

$A = A_0 + A_1 j$ implies that A^{-1} = $A_0^{-1} + A_1^{-1} j$. Now $A^{-1}(A^{-1})^{CT}$ = $(A_0^{-1} + A_1^{-1} j)$ $[(A_0^{-1})^{CT} - (A_1^{-1})^{CT} j)]$ = $(A_0^{CT} A_0)^{-1} - (A_1^{CT} A_1)^{-1} j$ = $(A_0^C A_0^T)^{-1} - (A_1^C A_1^T)^{-1} j$ [Since $A^T A^C = A^C A^T$] = $(A_0^{-1})^T (A_0^{-1})^C - (A_1^{-1})^T (A_1^{-1})^C j$ = $(A^{-1})^T (A^{-1})^C$.

Next we have to prove that λA is also a quaternion quasi-normal. That is $(\lambda A)(\lambda A)^{CT}$ = $(\lambda A)^T (\lambda A)^C$. Now, Let us consider, $(\lambda A)(\lambda A)^{CT}$ = $(\lambda A_0 + \lambda A_1 j)(\lambda A_0^{CT} - \lambda A_1^{CT} j)$ = $\lambda^2 (A_0 A_0^{CT}) - \lambda^2 (A_1 A_1^{CT}) j$ = $\lambda^2 (A_0^T A_0^C) - \lambda^2 (A_1^T A_1^C) j$ = $(\lambda A_0^T + \lambda A_1^T j)(\lambda A_0^C - \lambda A_1^C j)$ = $(\lambda A)^T (\lambda A)^C$.

Theorem: 4

If A and B are double representation of quaternion quasi-normal then $(A+B)$ and $(A-B)$ are also double representation of quaternion quasi-normal.

Proof:

Let us consider A and B are double representation of quaternion quasi-normal. Then $AA^{CT} = A^T A^C$ and $BB^{CT} = B^T B^C$. That is

$$A_0 A_0^{CT} - A_1 A_1^{CT} j \;=\; A_0^T A_0^C - A_1^T A_1^C j \qquad ....(1)$$

$$B_0 B_0^{CT} - B_1 B_1^{CT} j \;=\; B_0^T B_0^C - B_1^T B_1^C j \qquad ....(2)$$

Adding Eq.(1) and Eq.(2), we get

$$(A_0 A_0^{CT} - A_1 A_1^{CT} j) + (B_0 B_0^{CT} - B_1 B_1^{CT} j) \;=\; (A_0^T A_0^C - A_1^T A_1^C j) + (B_0^T B_0^C - B_1^T B_1^C j)$$

Therefore,

$$(A_0 A_0^{CT} + B_0 B_0^{CT}) - (A_1 A_1^{CT} + B_1 B_1^{CT}) j \;=\; (A_0^T A_0^C + B_0^T B_0^C) - (A_1^T A_1^C + B_1^T B_1^C) j \qquad ....(3)$$

We get $AB^{CT} + BA^{CT} \;=\; A^T B^C + B^T A^C$ ....(4)

(where $A = A_0 + A_1 j$ and $B = B_0 + B_1 j$).

$$\begin{aligned} AB^{CT} + BA^{CT} &= (A_0 + A_1 j)(B_0^{CT} - B_1^{CT} j) + (B_0 + B_1 j)(A_0^{CT} - A_1^{CT} j) \\ &= (A_0 B_0^{CT} + B_0 A_0^{CT}) - (A_1 B_1^{CT} + B_1 A_1^{CT}) j \qquad ....(5) \end{aligned}$$

$$\begin{aligned} A^T B^C + B^T A^C &= [(A_0^T + A_1^T j)(B_0^C - B_1^C j)] + [(B_0^T + B_1^T j)(A_0^C - A_1^C j)] \\ &= (A_0^T B_0^C + B_0^T A_0^C) - (A_1^T B_1^C + B_1^T A_1^C) j \qquad ....(6) \end{aligned}$$

Pre and post add by $AB^{CT} + BA^{CT} \;=\; A^T B^C + B^T A^C$ in equation Eq.(3).

$$\begin{aligned} &(AB^{CT} + BA^{CT}) + (A_0 A_0^{CT} + B_0 B_0^{CT}) - (A_1 A_1^{CT} + B_1 B_1^{CT}) j \;= \\ &\qquad (A^T B^C + B^T A^C) + (A_0^T A_0^C + B_0^T B_0^C) - (A_1^T A_1^C + B_1^T B_1^C) j \qquad ....(7) \end{aligned}$$

From Eq.(7), we get

$(AB^{CT} + BA^{CT}) + (A_0 A_0^{CT} + B_0 B_0^{CT}) - (A_1 A_1^{CT} + B_1 B_1^{CT}) j$ implies that

$$\begin{aligned} &= [A_0 B_0^{CT} + B_0 A_0^{CT} + A_0 A_0^{CT} + B_0 B_0^{CT}] - [A_1 B_1^{CT} + B_1 A_1^{CT} + A_1 A_1^{CT} + B_1 B_1^{CT}] j \\ &= [(A_0 + B_0)(A_0^{CT} + B_0^{CT})] - [(A_1 + B_1)(A_1^{CT} + B_1^{CT})] j \\ &= (A + B)[(A_0 + A_1 j)^{CT} + (B_0 + B_1 j)^{CT}] \end{aligned}$$

Therefore,

$$(AB^{CT} + BA^{CT}) + (A_0 A_0^{CT} + B_0 B_0^{CT}) - (A_1 A_1^{CT} + B_1 B_1^{CT}) j \;=\; (A + B)(A + B)^{CT} \qquad ....(8)$$

From Eq.(7), we get

$(A^T B^C + B^T A^C) + (A_0^T A_0^C + B_0^T B_0^C) - (A_1^T A_1^C + B_1^T B_1^C) j$ implies that

$$\begin{aligned} &= [A_0^T B_0^C + B_0^T A_0^C + A_0^T A_0^C + B_0^T B_0^C] - [A_1^T B_1^C + B_1^T A_1^C + A_1^T A_1^C + B_1^T B_1^C] j \\ &= [(A_0^T + B_0^T)(A_0^C + B_0^C)] - [(A_1^T + B_1^T)(A_1^C + B_1^C)] j \\ &= [(A_0 + A_1 j)^T + (B_0 + B_1 j)^T][(A_0 + A_1 j)^C + (B_0 + B_1 j)^C] \end{aligned}$$

Therefore,

$$(A^T B^C + B^T A^C) + (A_0^T A_0^C + B_0^T B_0^C) - (A_1^T A_1^C + B_1^T B_1^C) j \;=\; (A + B)^T (A + B)^C \qquad ....(9)$$

From Eq.(8) and Eq.(9) we get,

$(AB^{CT}+BA^{CT})+(A_0A_0^{CT}+B_0B_0^{CT})-(A_1A_1^{CT}+B_1B_1^{CT})j$ =
$$(A^TB^C+B^TA^C)+(A_0^TA_0^C+B_0^TB_0^C)-(A_1^TA_1^C+B_1^TB_1^C)j$$
Implies that, $(A+B)(A+B)^{CT}$ = $(A+B)^T(A+B)^C$.

Therefore, $(A+B)$ is double representation of quaternion quasi-normal. Similarly we can prove $(A-B)$ is double representation of quaternion quasi-normal.

Theorem: 5

If A be double representation of quaternion quasi-normal then AA^C is also double representation of quaternion quasi-normal.

Proof:

Given A be double representation of quaternion quasi-normal. Therefore $A=A_0+A_1j$. We have to prove AA^C is also double representation of quaternion quasi-normal. That is $(AA^C)(AA^C)^{CT}=(AA^C)^T(AA^C)^C$. We have $A=A_0+A_1j$ implies that $A^C=A_0^C-A_1^Cj$. Therefore, AA^C = $A_0A_0^C-A_1A_1^Cj$. $(AA^C)^{CT}$ = $A_0^TA_0^{CT}+A_1^TA_1^{CT}j$,$(AA^C)^T$ = $A_0^{CT}A_0^T-A_1^{CT}A_1^Tj$, $(AA^C)^C$ = $A_0^CA_0+A_1^CA_1j$.

From the above results, we get $(AA^C)(AA^C)^{CT}$ = $(A_0A_0^CA_0^TA_0^{CT})-(A_1A_1^CA_1^TA_1^{CT})j$ = $A_0(A_0A_0^{CT})A_0^{CT}-A_1(A_1A_1^{CT})A_1^{CT}j$ = $(A_0A_0^{CT})^2-(A_1A_1^{CT})^2j$. Now, $(AA^C)^T(AA^C)^C$ = $A_0^{CT}A_0^TA_0^CA_0-A_1^{CT}A_1^TA_1^CA_1j$ = $A_0^{CT}(A_0A_0^{CT})A_0-A_1^{CT}(A_1A_1^{CT})A_1j$ = $(A_0A_0^{CT})^2-(A_1A_1^{CT})^2j$ [since $A_0^{CT}A_0=A_0A_0^{CT}$] = $(A_0A_0^{CT})^2-(A_1A_1^{CT})^2j$

Therefore $(AA^C)(AA^C)^{CT}$ = $(AA^C)^T(AA^C)^C$. Therefore AA^C is also double representation of quaternion quasi-normal.

Remark:

We can easily proof that, A be double representation of quaternion quasi-normal then A^CA is also double representation of quaternion quasi-normal.

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